The Equality of Mixed Partials

Theorem: Suppose f:U→ℝ , where U is open in $\mathbb{R}^2$, is a function such that :

- $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist everywhere on U
- $\frac{\partial^2 f}{\partial y \partial x}$ exists everywhere on U and is continuous at p∈U
- $\frac{\partial^2 f}{\partial x \partial y}$ exists everywhere on U

Then $\left.\frac{\partial^2 f}{\partial x \partial y}\right|_p = \left.\frac{\partial^2 f}{\partial y \partial x}\right|_p$.

Proof:

Write $p=(x_0,y_0)$ and set $\Delta(h,k) = f(x_0+h, y_0+k)-f(x_0+h, y_0)-f(x_0, y_0+k) +f(x_0,y_0)$.

Set $G(x) = f(x, y_0+k) - f(x, y_0)$. Then $\Delta(h,k)=G(x_0+h) - G(x_0)$.

So $\Delta(h,k) = hG'(x)$ for some x between x_0 and x_0+h since G is differentiable with

$G'(x) = \left.\frac{\partial f}{\partial x}\right|_{(x,y_0+k)} - \left.\frac{\partial f}{\partial x}\right|_{(x,y_0)}$.

Now $G'(x)= k\left.\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right|_{(x,y)}$ for some y between y_0 and y_0+k since $\frac{\partial f}{\partial x}$ is differentiable as a function of y , x fixed.

Thus $\Delta(h,k) = hk\left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(x,y)}$.

In particular, $\lim_{(h,k)\to(0,0)} \frac{1}{hk}\Delta(h,k) = \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(x_0,y_0)}$ by continuity of $\frac{\partial^2 f}{\partial y \partial x}$ at (x_0,y_0).

Let $\left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(x_0,y_0)} = A$. And, given any $\varepsilon>0$, choose δ ∍ $|h|<\delta$, $|k|<\delta \Rightarrow$

$\left|\frac{1}{hk}\Delta(h,k) - A\right| < \varepsilon$.

Now $\left.\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right|_{(x_0,y_0)} - A = \lim_{h\to 0}\frac{1}{h}\left(\lim_{k\to 0}\frac{1}{k}(\Delta(h,k) - Ahk)\right)$ by definition of partial derivatives. But $\lim_{k\to 0}\left|\frac{1}{k}(\Delta(h,k) - Ahk)\right| \le \varepsilon|h|$ if $|h|<\delta$. So

$\left|\left.\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right|_{(x_0,y_0)} - A\right| \le \lim_{h\to 0}\frac{1}{|h|}\varepsilon|h| = \varepsilon$. Since this holds for all $\varepsilon>0$,

$\left.\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right|_{(x_0,y_0)} = A = \left.\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right|_{(x_0,y_0)}$ □